JOURNAL OF APPROXIMATION THEORY 35, 214–221 (1982)

A Cohen-Type Inequality for Jacobi Expansions and Divergence of Fourier Series on Compact Symmetric Spaces

B. DRESELER

Fachbereich Mathematik, Universität Siegen, D-5900 Siegen, West Germany

AND

P. M. SOARDI

Istituto Matematico, Universita di Milano, I-20133 Milano, Italy

Communicated by G. Meinardus

Received April 1, 1981

1. INTRODUCTION AND NOTATIONS

In a recent paper [4] we proved a Cohen-type inequality for ultraspherical series which implies divergence theorems for ultraspherical expansions and spherical harmonic expansions on the real sphere $S_{n-1} \subset \mathbb{R}^n$ with respect to arbitrary groupings of the degrees. In the following we extend the Cohen-type inequality to the larger class of Jacobi polynomials and we also get an estimate in the case of the critical endpoints of the Pollard interval $(\frac{4\alpha+4}{2\alpha+3}, \frac{4\alpha+4}{2\alpha+1})$ of L^p boundedness with respect to the natural grouping. Since the zonal spherical function on rank one symmetric spaces X of compact type can be viewed as Jacobi polynomials $R_m^{(\alpha,\beta)}$ with suitable $\alpha, \beta \geqslant -\frac{1}{2}$, Corollary 2 in [4] can be proved for this larger class of spaces. Let us mention that the proof of the divergence result in [10] is only correct for the case of the canonical grouping which is a corollary of an old theorem of Nicolaev [2]. A correct proof of the results in [10] follows from our Cohen-type inequality in the special case $p = 1$.

In the following we assume $\alpha \geqslant \beta \geqslant -1/2$, $\alpha > -1/2$. The results are completely analogous if $\alpha < \beta$. For x in $[-1, 1]$ let $R_m = R_m^{(\alpha,\beta)}$ be the Jacobi polynomial of degree m, defined by

0021-9045/82/070214–08$02.00/0

$$R_m^{(\alpha,\beta)}(x) = {}_2F_1[-m, m+\alpha+\beta+1; \alpha+1; (1-x)/2]$$
$$= \sum_{k=0}^{m} \frac{(-m)_k (m+\alpha+\beta+1)_k}{k!\,(\alpha+1)_k} \left(\frac{1-x}{2}\right)^k, \qquad (1.1)$$

where $(a)_k = \Gamma(k+a)/\Gamma(a)$. The Jacobi polynomials are orthogonal in $L^p_w = L^p_{w^{\alpha,\beta}}(-1,1)$, $1 \leqslant p < \infty$, where

$$\|f\|_p = \left\{ \int_{-1}^{1} |f(x)|^p\, w^{\alpha,\beta}(x)\, dx \right\}^{1/p}$$

and

$$w(x) = w^{\alpha,\beta}(x) = (1-x)^\alpha\, (1+x)^\beta.$$

As usual $\|f\|_\infty = \text{ess sup}_{x\in[-1,1]} |f(x)|$, and $f \in L^1_w$ has the expansion

$$f(x) \sim \sum_{m=0}^{\infty} c_m(f)\, h_m R_m$$

with

$$c_m(f) = \int_{-1}^{1} f(x)\, R_m(x)\, w(x)\, dx$$

and

$$h_m = h_m^{\alpha,\beta} = \left(\int_{-1}^{1} [R_m(x)]^2\, w(x)\, dx \right)^{-1}$$
$$= \frac{(2m+\alpha+\beta+1)\,\Gamma(m+\alpha+\beta+1)\,\Gamma(m+\alpha+1)}{2^{\alpha+\beta+1}\Gamma(m+\beta+1)\,\Gamma(m+1)\,\Gamma(\alpha+1)\,\Gamma(\alpha+1)}.$$

Denote by $*$ the usual convolution product in L^1_w [12]. For k in L^1_w let T: $L^p_w \to L^p_w$, $1 \leqslant p \leqslant \infty$, denote the convolutor $Tf = k * f$, $f \in L^p_w$, and set $|||k|||_p = \|T\|_p = \sup_{\|f\|_p=1} \|Tf\|_p$.

2. A Cohen-Type Inequality

THEOREM. *Let $n_1 < n_2 < \cdots < n_N$ be natural numbers and $c_{n_1}, c_{n_2}, \ldots, c_{n_N}$ be complex numbers with $|c_{n_N}| \geqslant 1$. Then*

$$\left\| \sum_{j=1}^{N} c_{n_j} h_{n_j} R_{n_j}^{(\alpha,\beta)} \right\|_p \geqslant M_{p,\alpha,\beta} \begin{cases} (n_N)^{(2\alpha+2)/p-(2\alpha+3)/2}, & 1 \leqslant p < \dfrac{4\alpha+4}{2\alpha+3}, \\ (\log n_N)^{1-1/p}, & p = \dfrac{4\alpha+4}{2\alpha+3}, \\ (\log n_N)^{1/p}, & p = \dfrac{4\alpha+4}{2\alpha+1}, \\ (n_N)^{(2\alpha+1)/2-(2\alpha+2)/p}, & \dfrac{4\alpha+4}{2\alpha+1} < p \leqslant \infty, \end{cases} \tag{2.1}$$

where $M_{p,\alpha,\beta}$ is a constant which only depends on p, α and β.

For the proof of the theorem we need the following three lemmas. Lemma 3 allows the reduction of the problem to good L^p norm estimates from below of the Jacobi polynomials $R_m^{(\alpha,\beta)}$. Lemmas1 and 2 give the L_p norm estimates.

LEMMA 1. *For each polynomial t_N on $\mathbb{R}$ of degree less than or equal to $N = 0, 1, 2,...,$ and $1 \leqslant p \leqslant q \leqslant \infty$ one has*

$$\| t_N \|_q \leqslant \text{const}_{p,q,\alpha}\, N^{(2\alpha+2)(1/p-1/q)} \| t_N \|_p. \tag{2.2}$$

LEMMA 2. *For $p_0 = (4\alpha + 4)/(2\alpha + 1)$ one gets*

$$\| R_m^{(\alpha,\beta)} \|_{p_0} \geqslant \text{const}_{\alpha,\beta} (\log m)^{1/p_0}\, m^{-\alpha-1/2}. \tag{2.3}$$

LEMMA 3. *There exists a sequence $(f_N^{\alpha,\beta})_{N=1}^{\infty}$ of continuous functions on $[-1, 1]$ with*

(i) $\|f_N^{\alpha,\beta}\|_\infty \leqslant 1$ $(N \in \mathbb{N})$,

(ii) $c_m(f_N^{\alpha,\beta}) = 0$ *for* $m \leqslant N$ $(m, N \in \mathbb{N})$,

(iii) $c_N(f_N^{\alpha,\beta}) \sim N^{-\alpha-1/2}$ $(N \to \infty)$.

Proof of Lemma 1. The proof is the same as the proof of Lemma 1 in [4] in the case $\alpha = \beta$. We include the proof then only for completeness.

Let $D_l^{\alpha,\beta}$ be the Dirichlet kernel $\sum_{m=0}^{l} h_m R_m$. Then one has $D_l^{\alpha,\beta} * t_l = t_l$ for each polynomial of degree less than or equal to l and by the inequality of Cauchy–Schwarz one gets

$$\| t_l \|_\infty \leqslant \| D_l^{\alpha,\beta} \|_2 \, \| t_l \|_2. \tag{2.4}$$

Choose r as the least even number greater than or equal to p. Then $t_N^{r/2}$ is a polynomial of degree less than or equal to $Nr/2$. From (2.4) we have

$$\|t_N^{r/2}\|_\infty \leqslant \|D_{rN/2}^{\alpha,\beta}\|_2 \left[\int_{-1}^{1} |t_N(x)|^r\, w(x)\, dx\right]^{1/2}$$
$$\leqslant \|D_{rN/2}^{\alpha,\beta}\|_2\, \|t_N\|_\infty^{(r-p)/2}\, \|t_N\|_p^{p/2}.$$

Since $h_m^{\alpha,\beta} \sim m^{2\alpha+1}$ $(m \to \infty)$ one gets

$$\|D_{rN/2}^{\alpha,\beta}\|_2 = \left(\sum_{m=0}^{rN/2} h_m\right)^{1/2} \leqslant \mathrm{const}_\alpha\, N^{\alpha+1}.$$

Thus we have proved (2.2) for $q = \infty$. Hence we get

$$\|t_N\|_q^q = \int_{-1}^{1} |t_N(x)|^{q-p}\, |t_N(x)|^p\, w(x)\, dx$$
$$\leqslant c_{p,q,\alpha} N^{(2\alpha+2)(1/p)(q-p)}\, \|t_N\|_p^{(q-p)}\, \|t_N\|_p^p$$

which implies (2.2).

Proof of Lemma 2. R. Askey informed the authors that Lemma 2 had been included by him as a problem in the fourth edition of Szegö's book [12, Problem 91]. Thus, we dropped the proof which follows from computations with Szegö's asymptotic formula [12, Theorem 8.21.13].

Proof of Lemma 3. Let M denote the least odd integer greater than or equal to $2\alpha+1$. Let $l = N+M$, $N = 1, 2, \ldots$. Define $f_N^{\alpha,\beta}\colon [-1, 1] \to \mathbb{R}$ by

$$f_N^{\alpha,\beta}(\cos\varphi) = \sin l\varphi\, \sin^{M-(2\alpha+1)}\frac{\varphi}{2}\, \cos^{M-(2\beta+1)}\frac{\varphi}{2}, \qquad 0 \leqslant \varphi \leqslant \pi.$$

Then (i) is clear and one has

$$c_m(f_N^{\alpha,\beta}) = \int_0^\pi R_m^{(\alpha,\beta)}(\cos\varphi)\, f_N^{\alpha,\beta}(\cos\varphi) \left(\sin\frac{\varphi}{2}\right)^{2\alpha+1} \left(\cos\frac{\varphi}{2}\right)^{2\beta+1} d\varphi$$
$$= \int_0^\pi R_m^{(\alpha,\beta)}(\cos\varphi)\, \sin l\varphi \left(\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\right)^M d\varphi$$
$$= \frac{1}{2^M}\int_0^\pi R_m^{(\alpha,\beta)}(\cos\varphi)\, \sin l\varphi\, \sin^M\varphi\, d\varphi.$$

Elementary computations show that

$$\sin l\varphi\, \sin^M\varphi = \sum_{s=l-M}^{l+M} q_s \cos s\varphi$$

with suitable coefficients q_s. In particular $q_{l-M} = 1/2(2i)^{M-1}$. Hence the lowest appearing frequency in the cosine expansion of $\sin l\varphi \sin^M \varphi$ is $N = l - M$. Now from (1.1) we get

$$R_m^{(\alpha,\beta)}(\cos\varphi) = \sum_{k=0}^{m} \frac{(-m)_k\,(m+\alpha+\beta+1)_k}{k!\,(\alpha+1)_k} \sin^{2k}\frac{\varphi}{2}.$$

An easy computation shows

$$\sin^{2k}\frac{\varphi}{2} = \frac{2}{(2i)^{2k}} \sum_{j=0}^{k} (-1)^j \binom{2k}{j} \cos(k-j)\,\varphi.$$

Hence

$$R_m^{(\alpha,\beta)}(\cos\varphi) = \frac{2}{(2i)^{2m}} \frac{(-m)_m\,(m+\alpha+\beta+1)_m}{m!\,(\alpha+1)_m} \cos m\varphi + \sum_{k=0}^{m-1} a_{k,m} \cos k\varphi.$$

This shows $c_m(f_N^{\alpha,\beta}) = 0$ for $m < N$ and

$$c_N(f_N^{\alpha,\beta}) = \mathrm{const}_M \frac{(-N)_N\,(N+\alpha+\beta+1)_N}{2^{2N} N!\,(\alpha+1)_N}.$$

From the duplication formula of the gamma function and the well-known behaviour of the gamma function at infinity one gets $c_N(f_N^{\alpha,\beta}) \sim N^{-\alpha-1/2}$.

Proof of the Theorem. One gets, with the testing function of Lemma 3 and $\tilde{N} = n_N$,

$$\begin{aligned}\left|\left|\left| \sum_{j=1}^{N} c_{n_j} h_{n_j} R_{n_j} \right|\right|\right|_p &\leqslant \left\| \sum c_{n_j} h_{n_j} R_{n_j} * f_{\tilde{N}}^{\alpha,\beta} \right\|_p \\ &= |c_{\tilde{N}}|\, h_{\tilde{N}}\, |c_{\tilde{N}}(f_{\tilde{N}}^{\alpha,\beta})|\, \|R_{\tilde{N}}\|_p \\ &\geqslant \mathrm{const}_{\alpha,\beta}\, \tilde{N}^{2\alpha+1} \tilde{N}^{-\alpha-1/2}\, \|R_{\tilde{N}}\|_p\end{aligned}$$

as $h_{\tilde{N}} \sim \tilde{N}^{2\alpha+1}$ and $|c_{\tilde{N}}| \geqslant 1$. We assume $\frac{4\alpha+4}{2\alpha+1} \leqslant p \leqslant \infty$. For $1 \leqslant p \leqslant \frac{4\alpha+4}{2\alpha+3}$ the assumption follows by duality. Now, for $p = \frac{4\alpha+4}{2\alpha+1}$ apply Lemma 2 and for $p > \frac{4\alpha+4}{2\alpha+1}$ apply Lemma 1 with $q = \infty$ and $t_{\tilde{N}} = R_{\tilde{N}}$. Recall that $\|R_{\tilde{N}}\|_\infty = R_{\tilde{N}}(1) = 1$.

Let us conclude this section with some remarks. In the special case $n_1 = 0, \ldots, n_N = N-1$, $c_0 = c_2 = \cdots = c_{N-1} = 1$ and $p = 1$ one can use the nice expression in [12, (4.5.3)] for the Dirichlet kernel $D_{N-1}^{\alpha,\beta}$ instead of Lemma 3. Let $c_{n_1} = \cdots = c_{n_{N-1}} = 0$ in the Theorem. This shows that in contrast to the situation on the torus ($\alpha = \beta = -1/2$) the growth of the convolutor norm of the Dirichlet kernel in our situation is at least the growth of the convolutor norm of the term $h_{n_N} R_{n_N}$ with the highest frequency n_N.

The estimate in Lemma 2 has been used by Newman and Rudin in [9] but they did not give a proof. The proof of Lemma 1 is analagous to the proof in the trigonometric case in Timan's book [13]. For $p=1$, $|c_{n_j}| \geqslant 1, j=1,..., N$ and $\alpha=-1/2$ inequality (2.1) is related to the famous conjecture of Littlewood [7] which has recently been proved [8].

3. Divergence Results

As in [4] we deduce here some divergence results that follow directly by usual technique from our Theorem. An increasing sequence $\{\Sigma_N\}$ of finite subsets Σ_N, $N=1,2,...,$ of $\mathbb{N}$ with $\bigcup_{N=1}^{\infty} \Sigma_N = \mathbb{N}$ is called a grouping on $\mathbb{N}$. On L_w^1 the partial sum operator relative to $\{\Sigma_N\}$ is defined by

$$S_N f = \sum_{m \in \Sigma_N} h_m R_m * f.$$

Corollary 1. *For $1 \leqslant p \leqslant \infty$ with $p \notin (\frac{4(\alpha+1)}{2\alpha+3}, \frac{4(\alpha+1)}{2\alpha+1})$ and for every grouping $\{\Sigma_N\}$ on $\mathbb{N}$ there exists a function $f \in L_w^p$ such that*

$$\limsup_{N \to \infty} \|S_N f\|_p = \infty.$$

For $\Sigma_N = \{0, 1,..., N\}$ and $p \in (P_0', P_0)$ Pollard [12] proved the convergence of the Jacobi expansions of L_w^p. Thus Corollary 1 completely settles the problem of divergence and extends the result of Newman and Rudin [9].

As mentioned in the Introduction, the zonal spherical functions on symmetric spaces X of compact type of rank one can be viewed as Jacobi polynomials (cf. [6]). The complete list of the spaces is S^m, $m \geqslant 2$, $P^m(\mathbb{R})$, $P^m(\mathbb{C})$, $P^m(\mathbb{H})$ and P^2 (Cayley). One has the following table. The zonal spherical function on X can be expressed by the Jacobi polynomials

TABLE I

X	$d = \dim X$	p	q
S^n	n	0	$d-1$
$P^n(\mathbb{R})$	n	$d-1$	0
$P^n(\mathbb{C})$	$2n$	$d-2$	1
$P^n(\mathbb{H})$	$4n$	$d-4$	3
P^2 (Cayley)	16	8	7

$R_m^{((d-2)/2,(q-1)/2)}$, $m \in \mathbb{N}$ and for $(\alpha, \beta) = (\frac{d-2}{2}, \frac{q-1}{2})$ the convolution structure of the Jacobi polynomials $R_m^{(\alpha,\beta)}$ is inherited from the corresponding symmetric space X. Denote by $L^p(X)$ the usual L^p space over X with respect to the normalized measure $d\omega$ associated with the volume element corresponding to the Riemanian structure on X. The convolutors S_N above extend to convolutors on $L^p(X)$ which are denoted by the same symbol.

COROLLARY 2. *Let X be a symmetric space of compact type and $d = \dim X$. Then for every p, $1 \leqslant p < 2d/(d+1)$, and for every choice of the grouping $\{\Sigma_N\}$ there exists a function $f \in L^p(X)$ such that*

$$\limsup_{N \to \infty} |S_N f(x)| = \infty$$

for $d\omega$ almost all x in X.

In the case $p = 1$ the result follows with well-known elementary arguments from the Theorem. Theorem 3 in [10] is a similar easy consequence of our Theorem with $p = 1$. In the case $1 < p < 2d/(d+1)$ Corollary 2 is a consequence of Corollary 1 and a result of Stein [11]. In contrast to Corollary 1, a much stronger result than Corollary 2 is known in the case of the canonical groupin $\Sigma_N = \{0, 1, ..., N\}$. In this case the result is true for $p < 2 = \lim_{d \to \infty} 2d/(d+1) = 2$. This follows from a result of Fefferman [5] combined with the Passage theorem of Bonami and Clerc [1] and the result of Stein [11] used above. Some of the results in this paper have recently been extended by the first author to compact symmetric spaces of arbitrary rank [3].

REFERENCES

1. A. BONAMI AND CLERC, Sommes de Cesàro et multiplicateurs des développements en harmoniques sphériques, *Trans. Amer. Math. Soc.* **183** (1973), 223–263.
2. E. W. CHENEY, "Introduction to Approximation Theory," McGraw–Hill, New York/St. Louis, 1966.
3. B. DRESELER, Norms of zonal spherical functions and Fourier series on compact symmetric spaces, *J. Funct. Anal.* **44** (1981), 74–86.
4. B. DRESELER AND P. M. SOARDI, A Cohen type inequality for ultraspherical series, *Arch. Math.*, to appear.
5. C. FEFFERMAN, The multiplier problem for the ball, *Ann. Math.* **94** (1971), 330–336.
6. R. GANGOLLI, Positive definite kernels on homogeneous spaces and certain stochastic processes related to Lévy's Brownian motion of several parameters, *Ann. Inst. H. Poincaré Sect. B* **3**, No. 2 (1967), 121–225.
7. G. HARDY AND J. LITTLEWOOD, A new proof of a theorem on rearrangements, *J. London Math. Soc.* **23** (1948), 163–168.

8. O. C. McGEHEE, L. PIGNO, AND B. SMITH, Hardy's inequality and the Littlewood conjecture, submitted.
9. J. NEW MAN AND W. RUDIN, Mean convergence of orthogonal series, *Proc. Amer. Math. Soc.* **3** (1952), 219–222.
10. W. SCHEMPP, Approximation on compact Riemannian globally symmetric manifolds of rank one, *J. Approx. Theory* **21** (1977), 236–245.
11. E. M. STEIN, On limits of sequences of operators, *Ann. Math.* **74** (1961), 140–170.
12. G. SZEGÖ, "Orthogonal Polynomials," AMS Coll. Publ., Providence, R.I., 1975.
13. A. F. TIMAN, "Theory of Approximation of Functions of a Real Variable," Macmillan, New York, 1963.